The Hessian matrix

The nullspace of the Hessian matrix in SIESTA is very important because it can lead to huge degeneracies of eigenvalues, and thus can give rise to huge contributions to the linear force.

For quadratic energies, the Hessian is essentially the coefficient of the second-order term (with a negative sign due to the conventions used in the code). For a spring potential, the Hessian would essentially be the spring constant. \( W = \frac{1}{2} \kappa \frac{d^2}{dx^2} \)

This Hessian is negative definite (has only negative eigenvalues) for a completely stable equilibrium.

The analytical nullspace

The nullspace of the Hessian matrix in SIESTA is very important because it can lead to huge displacements in directions that result in no change to the MHD force. It can easily be seen from the linearized ideal MHD equations that a plasma displacement that is purely parallel to the magnetic field everywhere will result in zero contribution to the linear force.

\[
\begin{align*}
\frac{\partial^2 F}{\partial \xi^2} &= \frac{1}{\mu_0} \left( \nabla \times \mathbf{B} \right) \cdot \nabla \mathbf{p} \\
\mathbf{B} &= \nabla \times \mathbf{B}
\end{align*}
\]

To find the nullspace, we want that all terms go to zero, thus \( \frac{\partial^2 F}{\partial \xi^2} = 0 \) implies that \( \mathbf{B} = 0 \). This is a nullspace of the ideal MHD force operator.

Hence, we would expect that a parallel plasma displacement would serve as a nullspace vector for the Hessian matrix, that is, any displacement should satisfy the equation:

\[
\begin{align*}
\mathbf{B} &= \nabla \times \mathbf{B} = 0 \\
\mathbf{F} &= \mathbf{B} \times \mathbf{p} = 0
\end{align*}
\]

Thus, we would expect that a parallel plasma displacement would serve as a nullspace vector for the Hessian matrix, that is, any displacement should satisfy the following equation:

\[
\frac{\partial^2 F}{\partial \xi^2} = \frac{\partial}{\partial \xi} \left( \frac{\partial^2 F}{\partial \xi^2} \right) = 0
\]

Put another way, we hope to find numerically that the nullspace eigenvectors of the SIESTA Hessian matrix are displacements that are essentially parallel to the magnetic field everywhere in the domain. This has been verified as discussed to the right.

Mechanical analogue for eigenproblem

A normal mode analysis on the equations of motion gives the following expression:

\[
\begin{align*}
\mathbf{F} &= \mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} \\
\mathbf{x} &= e^{in} \mathbf{k} \\
\dot{\mathbf{x}} &= e^{in} \mathbf{k} \omega
\end{align*}
\]

Thus we would expect that a parallel plasma displacement would serve as a nullspace of the ideal MHD force operator.

Nullspace structure for l=3 stellarator

Numerically, we are worried about the eigenproblem in which the inertia (I) matrix is set to unity:

In this case the eigenvalue problem is given by the equation to the right.

We want to solve for the eigenfrequencies and eigenmodes. Note that these are not the MHD normal modes as we have set the inertia matrix to the identity matrix. Real eigenfrequencies correspond to numerically stable modes while imaginary eigenfrequencies correspond to numerically unstable modes. For a classical, l=3 stellarator, a nullspace eigenvector was plotted below. This plasma displacement would result in no change in the linearized MHD force.

Plots of modulus of the nullspace eigenmode

High beta CDX-U stability analysis

A stability analysis was performed on a high beta (8%) CDX-U equilibrium that is known to be Mercier unstable. CDX-U (Current Drive Experiment-Upgrade) is a small tokamak located at Princeton Plasma Physics Lab.

Summary

- The nullspace eigenmodes of the SIESTA Hessian matrix have successfully been demonstrated to be parallel plasma displacements, agreeing with theory.
- A stability analysis has been performed on CDX-U, demonstrating the presence of unstable modes in an axisymmetric VMEC equilibrium. After convergence in SIESTA and formation of a magnetic island, the number of unstable modes decreases.
- The remaining instabilities are currently being considered. They could be due to numerical inaccuracies.

References